APPENDIX 8.2
(referred to in paragraph 8.21)

Arithmetic and geometric mean

1. This appendix considers the use of the arithmetic and geometric mean as measures of the average return on a financial asset, or index of returns on a portfolio of such assets, over a series of time periods.

2. The arithmetic mean is a simple average of percentage returns for each time period (where returns include both capital gains and dividends paid during that period).

3. The geometric mean is calculated from an index of returns (again reflecting both capital gains and dividends paid): if there are T periods of data, it is the Tth root of the value of the index at the end of the T periods divided by the value of the index at the start of the T periods, minus 1. The geometric mean therefore reflects the compound rate of return (excluding any transactions costs) that would have been earned by an investor who bought the shares in the index at the start of the T periods, reinvested all dividend income and rebalanced his/her portfolio in line with the index’s composition (a buy and hold strategy).

4. Thus, if an index starts at 100, falls to 80 and then increases again to 100, the arithmetic mean return is 2.5 per cent (the average of the initial 20 per cent fall and subsequent 25 per cent rise) and the geometric mean return is zero (because the value of the index at the end of the two periods is the same as at the beginning).

5. The arithmetic mean may differ according to the time period used: the arithmetic mean return calculated from annual data may be less than the arithmetic mean return calculated from monthly data (adjusted to annual equivalent basis). This is obviously not the case for the geometric mean, since it depends only on the returns index at the beginning and end of the T periods of data. The arithmetic mean will always exceed the geometric mean (unless the returns are exactly equal in each period when the two are equal): the greater the volatility of returns, the greater tends to be the difference between arithmetic and geometric means. If volatility falls over time, the arithmetic mean may also be expected to fall (other things remaining equal). Dimson, Marsh and Staunton suggest that their 6.6 per cent arithmetic mean equity premium, relative to Treasury Bills (shown in Table 8.1), should be adjusted to 5.9 per cent to reflect plausible estimates of early twenty-first century volatility, which they expect to be lower than the twentieth century average.

6. The extent of bias in the arithmetic and geometric means was considered by Blume under the assumption that returns were independently and normally distributed. Using simulation analysis, he found that the arithmetic mean overestimated and the geometric mean underestimated the true expected compound rate of return over N future periods, where N was greater than one but less than the T periods of data under consideration. Blume suggested that a suitable unbiased estimator could be calculated by taking a weighted average of the arithmetic and geometric means, with the weight on the arithmetic mean being \((T-N)/(T-1)\). Reflecting empirical evidence on the statistical properties of returns, Indro and Lee extended Blume’s analysis to include cases where returns were not independently and identically distributed. Their simulation results showed that Blume’s horizon-weighted average was less biased and more efficient than alternative estimates. The value of the horizon-weighted average depends on the length of the horizon but, for horizons of 10 to 30 years, will be closer to the arithmetic mean than the geometric mean shown in Table 8.1 (where T=100). For a horizon of ten years (suggested in paragraph 8.19) we calculate using Table 8.1 an ERP of 5.7 per cent relative to gilts and 6.4 per cent relative to Treasury Bills.

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3 The arithmetic mean would, however, be an unbiased estimator of the return over one period.
4 The weight on the geometric mean is one minus the weight on the arithmetic mean.
7. A different perspective is offered by Cooper\(^1\) who considered unbiased estimators of discount factors (ie \(1/(1 + r)^t\)) where ‘\(r\)’ is the true expected rate of return, which were relevant to capital budgeting applications. He found that the unbiased estimators of discount factors were generally much closer to the arithmetic mean of past annual returns than to the geometric mean.